CS103
Winter 2026

Lecture 05:

First-Order Logic

Part 2 of 2

First-Order Logic

Part 2

1. Announcements
2. Recap from Last Time
3. The Aristotelian Forms
4. The Art of Translation
5. Quantifier Ordering
6. Negating Quantified Statements
7. Restricted Quantifiers
8. Expressing Uniqueness
9. What's Next?

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Problem Set Two

- ***Problem Set One*** was due today at 1:00 PM.
  - You can extend the deadline to 1:00 PM Saturday using one of your late days. As usual, no late submissions will be accepted beyond 1:00 PM Saturday without prior approval.
- ***Problem Set Two*** goes out today. It's due next Friday at 1:00 PM.
  - Explore first-order logic!
  - Expand your proofwriting toolkit!

Readings for Problem Set Two

- We have some online readings for this problem set.
  - Check out the ***Guide to Logic Translations*** for more on how to convert from English to FOL.
  - Check out the ***Guide to Negations*** for information about how to negate formulas.
  - Check out the ***First-Order Translation Checklist*** for details on how to check your work.

Reminder: Stanford Honor Code

- As a reminder on course policies:
  - ChatGPT and other generative AI tools are off-limits for graded work.
  - You can discuss high-level ideas with other students, but can only share concrete solutions with your problem set partner.
- If you submitted something you shouldn't have, keep an eye out for the Regret Clause Form that will go out this weekend.
- ***We take the Honor Code seriously***. It promotes learning and basic fairness.

Hodgepodge

- Office hours have been updated! See course website.
- See syllabus for minor grading policy clarification.
- We have no classes on Monday, which is a holiday.
- The first Study Group Bonus is due Monday at 11:59 PM.

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What Is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates***
  - ***functions***
  - ***quantifiers***

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What Is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of obje…
- Augmen… …gic wit…

	… operate on …	… and produce
Connectives (↔, ∧, etc.) …	propositions	a proposition
Predicates (=, etc.) …	objects	a proposition
Functions …	objects	an object

What Is First-Order Logic?

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What Is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
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  - ***functions*** that map objects to one another, and
  - ***quantifiers***

What Is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about multiple objects.

Existential Quantifier

"Some bear is curious."

$$\exists b.\ (Bear(b) \wedge Curious(b))$$

∃ is the ***existential quantifier*** and says "there is a choice of b where the following is true."

Universal Quantifier

"For any natural number *n*,
n is even if and only if n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \rightarrow (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier*** and says "for all choices of *n*, the following is true."

Key Take-aways

“Some *P* is a *Q*”

translates as

$\exists x. (P(x) \land Q(x))$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

If x is an example, it *must* have property P on top of property Q.

Key Take-aways

"All *P*'s are *Q*'s"

translates as

$\forall x.\ (P(x) \to Q(x))$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

If x is a counterexample, it *must* have property P but not have property Q.

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The Aristotelian Forms

"All *A*s are *B*s"

$\forall x.\ (A(x) \to B(x))$

"Some *A*s are *B*s"

$\exists x.\ (A(x) \land B(x))$

"No *A*s are *B*s"

$\forall x.\ (A(x) \to \neg B(x))$

"Some *A*s aren't *B*s"

$\exists x.\ (A(x) \land \neg B(x))$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

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The Art of Translation

Using these predicates, write a first-order logic formula that says,

“Every person loves someone else.”

- Let *Person(p)* be the predicate, “*p* is a person.”
- Let *Loves*(*x*, *y*) be the predicate, “*x* loves *y*.”

Answer at

cs103.stanford.edu/pollev

The Art of Translation

Every person loves someone else

The Art of Translation

Every person loves some other person

The Art of Translation

Every person p loves some other person

The Art of Translation

Every person p loves some other person

"All *A*s are *B*s"

$\forall x. (A(x) \rightarrow B(x))$

The Art of Translation

$\forall p.\ (Person(p) \to$

p loves some other person

$)$

"All *A*s are *B*s"

$\boldsymbol{\forall x.\ (A(x) \to B(x))}$

The Art of Translation

$\forall p.\ (Person(p) \rightarrow$

p loves some other person

$)$

The Art of Translation

$\forall p.\ (Person(p) \to$

there is some other person that p loves

$)$

The Art of Translation

$\forall p. \ (Person(p) \rightarrow$

there is a person other than p that p loves

$)$

The Art of Translation

$\forall p.\ (Person(p) \rightarrow$

there is a person q, other than p, where p loves q

$)$

The Art of Translation

$\forall p. \; (Person(p) \rightarrow$

there is a person q, other than p, where

p loves q

$)$

The Art of Translation

$\forall p. \ (Person(p) \rightarrow$

there is a person q, other than p, where

p loves q

$)$

"Some *A*s are *B*s"

$\exists x. \ (A(x) \land B(x))$

The Art of Translation

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q)\ \land$ *, other than p, where*

p loves q

$\quad)$

$)$

"Some *A*s are *B*s"

$\boldsymbol{\exists x.\ (A(x) \land B(x))}$

The Art of Translation

$\forall p. \ (Person(p) \rightarrow$

$\quad \exists q. \ (Person(q) \land$ *, other than p, where*

p loves q

$\quad)$

$)$

The Art of Translation

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q) \land p \neq q \land$

$\quad\quad$ *p loves q*

$\quad)$

$)$

The Art of Translation

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

The Art of Translation

Using these predicates, write a first-order logic formula that says,

"There is a person whom everyone else loves."

- Let *Person(p)* be the predicate, "*p* is a person."
- Let *Loves*(*x*, *y*) be the predicate, "*x* loves *y*."

Answer at

cs103.stanford.edu/pollev

The Art of Translation

There is a person whom everyone else loves

The Art of Translation

There is a person p where everyone else loves p

The Art of Translation

There is a person p where everyone else loves p

"Some *A*s are *B*s"

$\exists x.\ (A(x) \land B(x))$

The Art of Translation

$\exists p. \, (Person(p) \land$

everyone else loves p

$)$

"Some *A*s are *B*s"

$\boldsymbol{\exists x. \, (A(x) \land B(x))}$

The Art of Translation

$\exists p. \ (Person(p) \ \land$

everyone else loves p

$)$

The Art of Translation

$\exists p. \; (Person(p) \; \wedge$

every other person q loves p

$)$

The Art of Translation

$\exists p. \ (Person(p) \ \wedge$

every person q, other than p, loves p

$)$

The Art of Translation

$\exists p.\ (Person(p)\ \land$

every person q, other than p, loves p

$)$

"All *A*s are *B*s"

$\forall x.\ (A(x) \rightarrow B(x))$

The Art of Translation

$\exists p.\ (Person(p)\ \land$

$\quad \forall q.\ (Person(q)\ \land\ p \neq q\ \rightarrow$

$\quad\quad$ *q loves p*

$\quad)$

$)$

"All *A*s are *B*s"

$\forall x.\ (A(x) \rightarrow B(x))$

The Art of Translation

$\exists p. \ (Person(p) \land$

$\quad \forall q. \ (Person(q) \land p \neq q \rightarrow$

q loves p

$\quad)$

$)$

The Art of Translation

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\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
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\end{aligned}
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Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.

"Every person loves someone else"

For every person… $\forall p.\ (Person(p) \to$

… there is another person … $\exists q.\ (Person(q) \land p \neq q \land$

… they love $Loves(p, q)$

$)$

$)$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.

"There is someone everyone else loves."

There is a person…

… that everyone else …

… loves.

$$\exists p.\ (Person(p) \land$$
$$\quad \forall q.\ (Person(q) \land p \neq q \rightarrow$$
$$\quad\quad Loves(q, p)$$
$$\quad)$$
$$)$$

For Comparison

For every person…

… there is another person …

… they love

$$\forall p.\ (Person(p) \to \\ \quad \exists q.\ (Person(q) \land p \neq q \land \\ \qquad Loves(p, q) \\ \quad) \\ )$$

There is a person…

… that everyone else …

… loves.

$$\exists p.\ (Person(p) \land \\ \quad \forall q.\ (Person(q) \land p \neq q \to \\ \qquad Loves(q, p) \\ \quad) \\ )$$

Quantifier Ordering

- Consider these two first-order formulas:

$$\forall m.\ \exists n.\ m < n.$$

$$\exists n.\ \forall m.\ m < n.$$

- Pretend for the moment that our world consists purely of natural numbers, so the variables *m* and *n* refer specifically to natural numbers.
- One of these statements is true. The other is false.
- Which is which?
- Why?

Answer at

cs103.stanford.edu/pollev

Quantifier Ordering

- Consider these two first-order formulas:

$$\forall m.\ \exists n.\ m < n.$$

$$\exists n.\ \forall m.\ m < n.$$

- This says

for every natural number *m*,
there's a larger natural number *n*.

- This is true: given any $m \in \mathbb{N}$, we can choose n to be $m + 1$.
- Notice that we can pick n based on m, and we don't have to pick the same n each time.

Quantifier Ordering

- Consider these two first-order formulas:

$$\forall m.\ \exists n.\ m < n.$$

$$\exists n.\ \forall m.\ m < n.$$

- This says

there is a natural number *n*
that's larger than every natural number *m*

- This is false: no natural number is bigger than every natural number.
- Because $\exists n$ comes first, we have to make a single choice of n that works regardless of what we choose for m.

Quantifier Ordering

- The statement

  $$\forall x.\ \exists y.\ P(x, y)$$

  means “for any choice of x, there's some choice of y where $P(x, y)$ is true.”
- The choice of y can be different every time and can depend on x.

Quantifier Ordering

- The statement

  $$\exists x.\ \forall y.\ P(x, y)$$

  means "there is some *x* where for any choice of *y*, we get that *P*(*x*, *y*) is true."
- Since the inner part has to work for any choice of *y*, this places a lot of constraints on what *x* can be.

Key Take-away

Order matters when mixing existential and universal quantifiers!

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An Extremely Important Table

	When is this **true**?	When is this **false**?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	There is an x where $P(x)$ is false.
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	For all objects x, $P(x)$ is false.
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	There is an x where $P(x)$ is true.
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

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$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	There is an x where $P(x)$ is true.
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	When is this **true**?	When is this **false**?
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$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	**For all objects *x*, *P*(*x*) is false.**	There is an x where $P(x)$ is true.
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this **true**?	When is this **false**?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
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An Extremely Important Table

	When is this **true**?	When is this **false**?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	**There is an *x* where *P*(*x*) is true.**	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	**There is an *x* where *P*(*x*) is true.**
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this **true**?	When is this **false**?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	**There is an x where $P(x)$ is true.**	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this **true**?	When is this **false**?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this **true**?	When is this **false**?
$\forall x.\ P(x)$	**For all objects *x*, *P*(*x*) is true.**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	**For all objects *x*, *P*(*x*) is true.**

An Extremely Important Table

	When is this **true**?	When is this **false**?
$\forall x.\ P(x)$	**For all objects x, $P(x)$ is true.**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	$\forall x.\ P(x)$

An Extremely Important Table

	When is this **true**?	When is this **false**?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	$\forall x.\ P(x)$

Negating First-Order Statements

- Use the equivalences

  $$\neg\forall x.\ A \quad \textit{is equivalent to} \quad \exists x.\ \neg A$$

  $$\neg\exists x.\ A \quad \textit{is equivalent to} \quad \forall x.\ \neg A$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

Taking a Negation

$$\forall x.\ \exists y.\ Loves(x, y)$$

("Everyone loves someone.")

$$\neg\forall x.\ \exists y.\ Loves(x, y)$$

$$\exists x.\ \neg\exists y.\ Loves(x, y)$$

$$\exists x.\ \forall y.\ \neg Loves(x, y)$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

$$\neg(p \land q) \quad \textit{is equivalent to} \quad p \to \neg q$$

$$\neg(p \to q) \quad \textit{is equivalent to} \quad p \land \neg q$$

- These identities are useful when negating statements involving quantifiers.
  - $\land$ is used in existentially-quantified statements.
  - $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we ***strongly recommend*** using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

Negating Quantifiers

- What is the negation of the following statement, which says "there is a cute puppy"?

$$\exists x.\ (\textit{Puppy}(x) \land \textit{Cute}(x))$$

Answer at

cs103.stanford.edu/pollev

Negating Quantifiers

- What is the negation of the following statement, which says "there is a cute puppy"?

$$\boldsymbol{\exists x. (Puppy(x) \land Cute(x))}$$

- We can obtain it as follows:

$$\neg\exists x. (Puppy(x) \land Cute(x))$$

$$\forall x. \neg(Puppy(x) \land Cute(x))$$

$$\forall x. (Puppy(x) \to \neg Cute(x))$$

- This says "no puppy is cute."
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

Negating Quantifiers

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

(*"There is a set with no elements."*)

$$\neg\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

$$\forall S.\ \neg(Set(S) \land \forall x.\ \neg x \notin S)$$

$$\forall S.\ (Set(S) \to \neg\forall x.\ x \notin S)$$

$$\forall S.\ (Set(S) \to \exists x.\ \neg(x \notin S))$$

$$\forall S.\ (Set(S) \to \exists x.\ x \in S)$$

(*"Every set contains at least one element."*)

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Quantifying Over Sets

- The notation

  $$\forall x \in S.\ P(x)$$

  means "for any element x of set S, $P(x)$ holds." (It's vacuously true if S is empty.)

- The notation

  $$\exists x \in S.\ P(x)$$

  means "there is an element x of set S where $P(x)$ holds." (It's false if S is empty.)

Quantifying Over Sets

- The syntax

$$\forall x \in S.\ P(x)$$

$$\exists x \in S.\ P(x)$$

  is allowed for quantifying over sets.

- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.
- For example, don't do things like this:

⚠️ $\forall x \text{ with } P(x).\ Q(x)$ ⚠️

⚠️ $\forall y \text{ such that } P(y) \land Q(y).\ R(y).$ ⚠️

⚠️ $\exists P(x).\ Q(x)$ ⚠️

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Expressing Uniqueness

Using this predicate, write a first-order logic formula that says,

"There is only one way to find out."

- Let *WayToFindOut(p)* be the predicate, "*w* is a way to find out."

Expressing Uniqueness

There is only one way to find out.

Expressing Uniqueness

Something is a way to find out, and nothing else is.

Expressing Uniqueness

Some thing w is a way to find out, and nothing else is.

Expressing Uniqueness

Some thing w is a way to find out, and nothing besides w is a way to find out

Expressing Uniqueness

$\exists w. \ (WayToFindOut(w) \ \land$

nothing besides w is way to find out

$)$

Expressing Uniqueness

$\exists w.\ (WayToFindOut(w)\ \land$

anything that isn't w isn't a way to find out

$)$

Expressing Uniqueness

$\exists w. \ (WayToFindOut(w) \ \land$

any thing x that isn't w isn't a way to find out

$)$

Expressing Uniqueness

$\exists w.\ (\mathit{WayToFindOut}(w) \land$
$\quad \forall x.\ (x \neq w \rightarrow$ *x isn't a way to find out*$)$
$)$

Expressing Uniqueness

$$\exists w.\ (\mathit{WayToFindOut}(w) \land$$
$$\quad \forall x.\ (x \neq w \rightarrow \neg \mathit{WayToFindOut}(x))$$
$$)$$

Expressing Uniqueness

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\quad \forall x.\ (x \neq w \to \neg \mathit{WayToFindOut}(x)) \\
&)
\end{aligned}
$$

Expressing Uniqueness

$$\exists w.\ (\mathit{WayToFindOut}(w)\ \land$$
$$\quad \forall x.\ (\mathit{WayToFindOut}(x) \rightarrow x = w)$$
$$)$$

Expressing Uniqueness

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\quad \forall x.\ (\mathit{WayToFindOut}(x) \to x = w) \\
&)
\end{aligned}
$$

Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that
  - there exists at least one object with that property, and that
  - there are no other objects with that property.
- You sometimes see a special “uniqueness quantifier” used to express this:

$$\exists! x.\ P(x)$$

- For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular ∀ and ∃ quantifiers.

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Next Time

- ***Functions***
    - How do we model transformations and pairings?
- ***First-Order Definitions***
    - Where does first-order logic come into all of this?
- ***Proofs with Definitions***
    - How does first-order logic interact with proofs?